The Weibull Distribution
Parameter Estimation - A Case Study

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In Part I we developed the mathematics for the Weibull distribution. In Part II we will develop the mathematics to estimate the values of the Weibull distibution parameters κ (shape parameter) and λ (scale parameter). To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

Data from the BLS shows that approximately 20% of new businesses fail during the first two years of being open, 45% during the first five years, and 65% during the first 10 years. Only 25% of new businesses make it to 15 years or more.

Using the BLS data the survival rates by cohort are...

Table 1: Survival Rates

Symbol	Year	Failure Rate	Survival Rate
t_1	2	0.20	0.80
t_2	5	0.45	0.55
t_3	10	0.65	0.35
t_4	15	0.75	0.25

Our task is to answer the following question...

Question 1: What are the parameter estimates for a Weibull distribution?

Question 2: Graph the Weibull distribution?

Question 3: What is the probability that a company will survive 20 years or more?

Question 4: How well does the parameterized Weibull distribution match the actual data?

The Mathematics

We will define the variable t to be time in years and the function $S(t)$ to be the survival rate over the time interval $[0, t]$, which is the probability that the failure event will arrive sometime after time t. The equation for the survival rate of a Weibull-distributed random variable t is... [1]

$$S(t) = \text{Prob}\Bigg[\text{Failure event arrival time} \geq t\Bigg] = \text{Exp}\left\{ -\left(\frac{1}{\lambda}\,t\right)^{k}\right\} \tag{1}$$

The survival rate in Equation (1) above is a function of the parameters κ and λ, both of which must be greater than zero. Our job in this white paper is to estimate the values of those two parameters.

Using Appendix Equation (14) below and solving the survival function (Equation (1) above) for the random variable t we get the folloing equality...

$$\frac{1}{\lambda}\left[-\ln\bigg(S(t)\bigg)\right]^{\frac{1}{\kappa}} = t \tag{2}$$

We will estimate the two Weibull distribution parameters by setting κ and λ to values that exactly match (i.e. anchor) the two end points of the survival rate curve as defined in Table 1 above. Per our hypothetical probalem the survival rates at the beginning and end of the curve are...

$$t_1 = 0.80 \text{ ...and... } t_4 = 0.25 \tag{3}$$

Using Equations (2) and (3) above, to calculate our two parameter values we need to solve the following two simultaneous equations...

$$\frac{1}{\lambda}\left[-\ln\left(S(t_1)\right)\right]^{\frac{1}{\kappa}} = t_1 \text{ ...and... } \frac{1}{\lambda}\left[-\ln\left(S(t_4)\right)\right]^{\frac{1}{\kappa}} = t_4 \tag{4}$$

If we divide the first equation by the second equation then we get the following equation to solve via the Newton Raphson method for solving non-linear equations...

$$\frac{[-\ln(S(t_1))]^{\frac{1}{\kappa}}}{[-\ln(S(t_4))]^{\frac{1}{\kappa}}} = \frac{t_1}{t_4} \tag{5}$$

Note that by taking the ratios of the two equations in Equation (4) above we eliminate the unknown parameter λ, which leaves us with only the parameter κ to solve for.

Using Appendix Equation (16) below we will make the following function definitions...

$$g(\hat{\kappa}) = \left[-\ln(S(t_1))\right]^{1/\hat{\kappa}} \text{ ...and... } g'(\hat{\kappa}) = -\left[-\ln\left(S(t_1)\right)\right]^{1/\hat{\kappa}} \ln\left(-\ln\left(S(t_1)\right)\right)\hat{\kappa}^{-2} \tag{6}$$

$$h(\hat{\kappa}) = \left[-\ln(S(t_4))\right]^{1/\hat{\kappa}} \text{ ...and... } h'(\hat{\kappa}) = -\left[-\ln\left(S(t_4)\right)\right]^{1/\hat{\kappa}} \ln\left(-\ln\left(S(t_4)\right)\right)\hat{\kappa}^{-2} \tag{7}$$

Using Equations (5), (6) and (7) above we will make the following function definitions...

$$f(\kappa) = \frac{t_1}{t_4} \text{ ...and... } f(\hat{\kappa}) = \frac{g(\hat{\kappa})}{h(\hat{\kappa})} \text{ ...and... } f'(\hat{\kappa}) = \frac{g'(\hat{\kappa})\,h(\hat{\kappa}) - h'(\hat{\kappa})\,g(\hat{\kappa})}{h(\hat{\kappa})^2} \tag{8}$$

Using Equations (6), (7) and (8) above we can rewrite Equation (5) as...

$$\frac{g(\hat{\kappa})}{h(\hat{\kappa})} = \frac{t_1}{t_4} \tag{9}$$

We will solve Equation (9) above for the parameter κ by iterating the following Newton Raphson equation until the error term $\epsilon \approx 0$... [2]

$$\hat{\kappa} + \frac{f(\kappa) - f(\hat{\kappa})}{f'(\hat{\kappa})} = \kappa + \epsilon \tag{10}$$

Using the value of the parameter κ derived from Equation (10) above the equation for the value of the parameter λ is...

$$\text{if... } S(t) = \text{Exp}\left\{-\left(\frac{1}{\lambda}\,t\right)^k\right\} \text{ ...then... } \lambda = \left\{\left[-\ln\left(S(t_1)\right)^{\frac{1}{\kappa}}\right]\Big/\,t\right\}^{-1} \tag{11}$$

The Answers To Our Hypothetical Problem

Question 1: What are the parameter estimates for a Weibull distribution?

Using Equation (10) above the value of the parameter κ is...

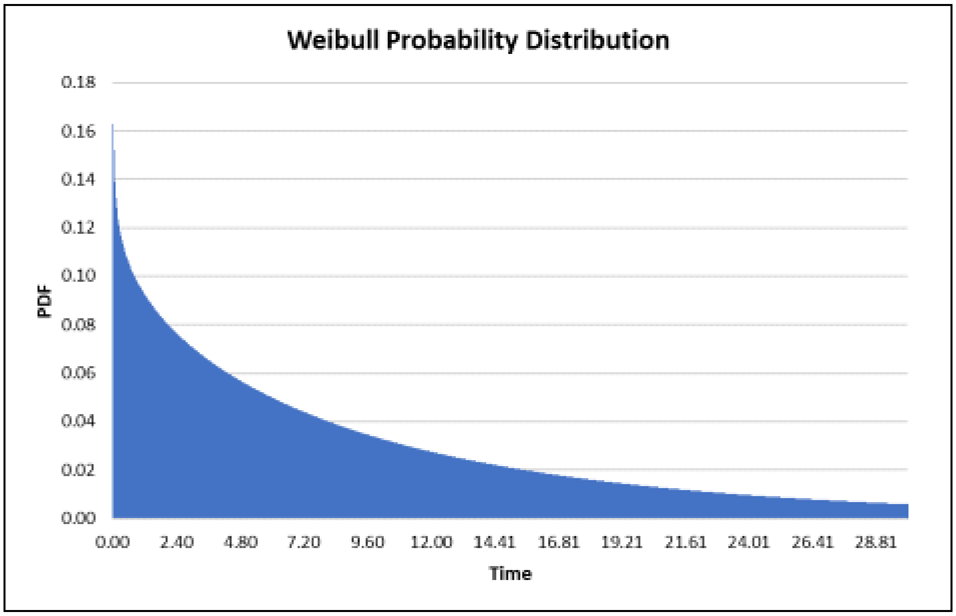
Iteration	$\hat{\kappa}$	$f(\kappa)$	$f(\hat{\kappa})$	$f'(\hat{\kappa})$	new $\hat{\kappa}$
1	0.5000	0.1333	0.0259	0.1893	1.0675
2	1.0675	0.1333	0.1807	0.2896	0.9040
3	0.9040	0.1333	0.1326	0.2963	0.9065
4	0.9065	0.1333	0.1333	0.2964	0.9065
5	0.9065	0.1333	0.1333	0.2964	0.9065

Per the Newton Raphson method for solving non-linear equations the variable $\kappa = 0.9065$

Using Equation (11) above and the calculated value of κ (0.9065) the value of the parameter λ is...

$$\lambda = \left\{ \left[-\ln\left(0.80\right)^{1/0.9065} \right] \Big/ 2.00 \right\}^{-1} = 10.46 \tag{12}$$

Question 2: Graph the Weibull distribution?



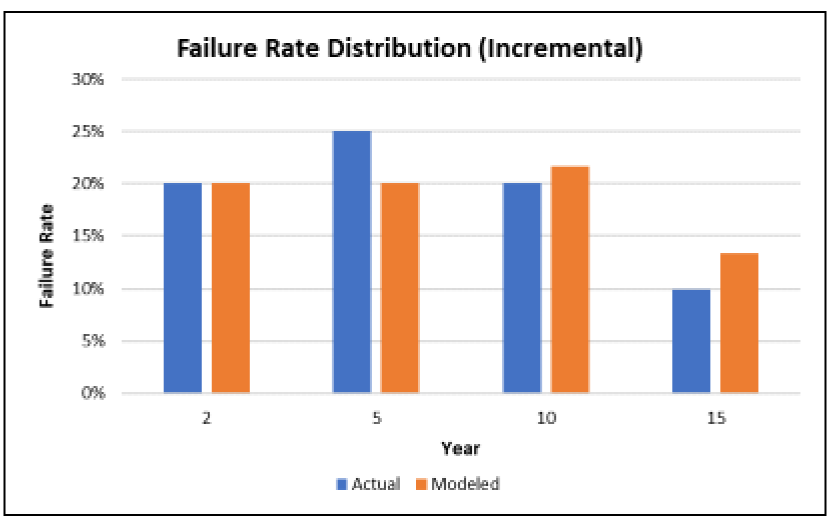
Note that since the value of $\kappa < 1.00$ the graph above approximates the exponential distribution.

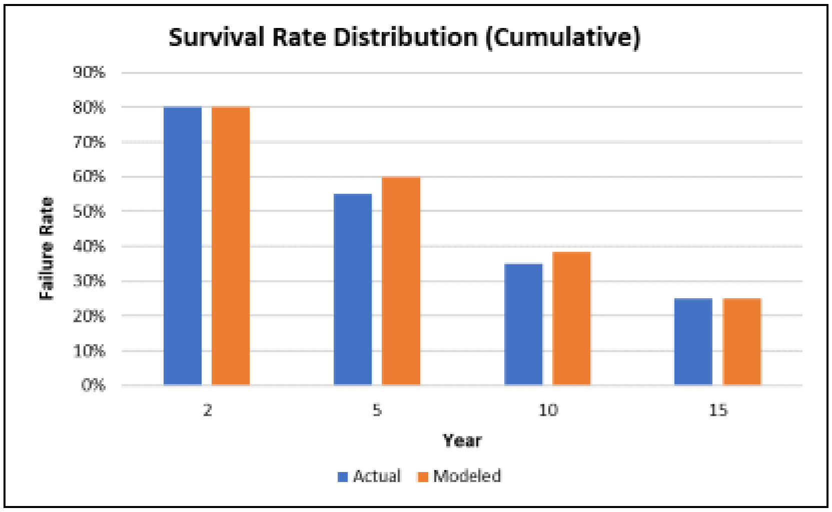
Question 3: What is the probability that a company will survive 20 years or more?

Using Equation (1) above the answer to the question is...

$$S(t) = 1 - \text{Exp}\left\{ -\left(\frac{1}{10.46} \times 2.00\right)^{0.9065} \right\} = 0.1740 \tag{13}$$

Question 4: How well does the parameterized Weibull distribution match the actual data?





References

[1] Gary Schurman, *The Weibull Distribution - The Mathematics,* December, 2022.

[2] Gary Schurman, *The Newton-Raphson Method For Solving Nonlinear Equations,* October, 2009.

Appendix

A. Taking the log of equation (1) above and solving for the random variable t...

$$\text{if...}\;\; \ln\bigg(S(t)\bigg) = -\bigg(\frac{1}{\lambda}\,t\bigg)^{\kappa} \;\;\text{...then...}\;\; t = \frac{1}{\lambda}\bigg[-\ln\bigg(S(t)\bigg)\bigg]^{\frac{1}{\kappa}} \tag{14}$$

B. The derivative of the following equation is...

$$\frac{\delta}{\delta x}\,a^{\frac{1}{x}} = -\frac{a^{\frac{1}{x}}\ln(a)}{x^2} \tag{15}$$

C. Using Equation (15) above as our guide, the derivative of the following equation is...

$$\frac{\delta}{\delta\hat{\kappa}}\bigg[-\ln\bigg(S(t)\bigg)\bigg]^{1/\hat{\kappa}} = -\bigg[-\ln\bigg(S(t)\bigg)\bigg]^{1/\hat{\kappa}}\ln\bigg(-\ln\bigg(S(t)\bigg)\bigg)\,\hat{\kappa}^{-2} \tag{16}$$